Rose-Hulman Undergraduate Mathematics Journal

Volume 3
Issue 2

Article 4

Investigations into the Kaprekar Process

Robert Ellis
East Tennessee State University

Jason Lewis
East Tennessee State University, jasonlewis_tn@yahoo.com

Follow this and additional works at: https://scholar.rose-hulman.edu/rhumj

Recommended Citation
Ellis, Robert and Lewis, Jason (2002) "Investigations into the Kaprekar Process," *Rose-Hulman Undergraduate Mathematics Journal*: Vol. 3 : Iss. 2 , Article 4.
Available at: https://scholar.rose-hulman.edu/rhumj/vol3/iss2/4

Investigations into the Kaprekar Process

Robert W. Ellis
Department of Mathematics
East Tennessee State University
Johnson City, TN, USA
rellis01@utk.edu

Jason R. Lewis
Department of Mathematics
East Tennessee State University
Johnson City, TN, USA
jasonlewis_tn@yahoo.com

October 27, 2002

1. Introduction

In 1946 Kaprekar proposed an interesting mathematical problem. His problem, through which emerges what later became known as the Kaprekar Constant, is as follows:

Given any four digit number, *abcd*, within seven iterations of the Kaprekar Process, you will reach the number 6174. The Kaprekar Process is as follows:

(1) Take any four-digit number, *abcd*, consisting of four distinct digits.

(2) Rearrange the digits into descending order and then into ascending order, including the leading zeros.

(3) Subtract the latter from the former.

(4) Repeat this procedure using the result from step 3, using the leading zeros, until the number 6174 is reached [1].

Once the number 6174 is reached, it will regenerate itself. For example, take the number 5643.

1	6543	-	3456	=	3087
2	8730	-	0378	=	8352
3	8532	-	2358	=	6174
	7641	-	1467	=	6174

The number 5643 reached 6174 in three iterations.

It took Kaprekar nearly three years to arrive at the proof that this procedure works for all four-digit numbers, that are not for multiples of 1111 [2].

Once we finished our study of the Kaprekar Process with four-digit numbers, we decided to explore what would happen if one applied the Kaprekar Process to numbers with different digit lengths (number of digits). In this paper, we will answer this question and a few others that came up in the course of our research.

Our major contributions to this problem include (1) a statistical analysis of this process on five-digit numbers, (2) further investigation and the discovery of relationships between four-digit numbers after application of the Kaprekar Process, (3) a summary of results of the Kaprekar Process arranged by digit length, and, last but not least, (4) the interesting palindromic sequences which have become the *leitmotif* of this research.

2. Kaprekar Procedure with Four-Digit Numbers

In this section, we discuss the Kaprekar Process when it is applied to four-digit numbers. One of the remarkable things that we discovered is that after one iteration of the Kaprekar Process, all the numbers generated will be divisible by nine. This led us to the following theorem.

Theorem 2.1. Consider any four-digit number, consisting of the digits a, b, c, and d, where all four digits are not equal. Once the digits are arranged in descending order, if, without loss of generality, $a \geq b = c \geq d$, then the sum of the digits after one iteration of the Kaprekar Process will be 27, which is divisible by nine. If, without loss of generality, $a \geq b > c \geq d$, then the sum of the digits after one iteration of the Kaprekar Process will be 18, which is also divisible by nine.

PROOF:

$b = c$

	a	b	b	d
-	d	b	b	a
	a-1-d	9	9	d+10-a

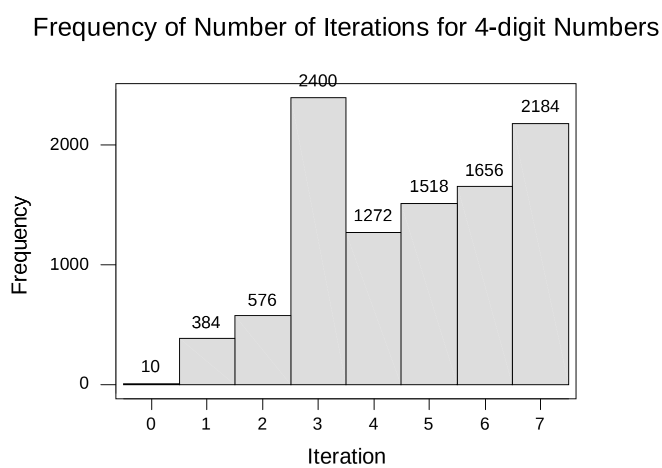
$a-1-d+9+9+d+10-a=27$

		b	*c*	
	a	*b*	*c*	*d*
-	*d*	*c*	*b*	*a*
	$a-d$	$b-1-c$	$c+9-b$	$d+10-a$

$$a-d+b-1-c+c+9-b+d+10-a=18$$

After analyzing all 10,000 four-digit numbers, and determining the path that each one of them takes to reach the number 6174 in the Kaprekar Process, we were able to perform a statistical analysis on the results generated. The following is a summary of the statistical analysis; the maximum number of iterations required to reach 6174 is seven, the mode of the number of iterations is three, the average number of iterations is 4.6, and the standard deviation is 1.8. Figure 1 is a histogram of the frequencies of the number of iterations for four-digit numbers. Here we use the convention that the numbers 0000, 1111, …, 9999 take zero iterations, but the number 6174 takes one iteration for the process to terminate

Figure 1



Consider any four-digit number, *abcd*, consisting of four distinct digits arranged in descending order. The table in Figure 2 can be used to determine the number of iterations required to reach 6174 using the Kaprekar Process. To determine the required number of iterations to reach 6174, you first subtract *d* from *a* and then *c* from *b*. Then find where these two numbers intersect on the table in Figure 2. That number will be the required number of iterations to reach 6174. Throughout our research we have been unable to find a similar result.

Figure 2
The Required Number of Iterations to Reach 6174

a – *d*										
9	4	4	3	3	7	7	7	3	3	4
8	6	3	6	5	2	7	2	5	6	
7	4	3	5	6	3	5	3	6		
6	6	7	1	3	4	5	4			
5	6	7	7	5	5	5				
4	4	7	2	3	4					
3	6	3	5	6						
2	4	3	6							
1	5	4								
0	0									
	0	1	2	3	4	5	6	7	8	9

b – *c*

One interesting fact that we discovered in our research leading to Figure 2 is that it contains many palindromes. All of the diagonals form palindromes (when you exclude the zero column and the zero row), with the exception of the fifth diagonal (7-1-6-2-7). Also, the columns and rows produce their own palindromes. For the columns, the palindromes start at the row with the value 10-*b*-c (excluding column 0 which starts at the ninth row), and goes all the way to the last number in the column. For example, in the fourth column (*b*-*c*=3), the palindrome starts at the row with a value of seven, and therefore the palindrome is 63536. The only exception to this is the third column. One can use a similar technique to find the palindromes in the rows. We have yet to discover the meaning of these palindromes (if there is one), but we feel that it is not mere coincidence.

Once we had determined the path that each four-digit number takes to reach Kaprekar's Constant of 6174, we were able to construct a family tree of the distinct paths. Before we explain the family tree, we must first state some definitions.

Species: Each four-digit number (e.g. 5643).

Base Number: A number with its digits in descending order (e.g. 6543).

Order: All base numbers that share the same path to 6174.

Class: All orders that require the same number of iterations to reach 6174.

Kingdom: The set of all the four-digit numbers.

Family Tree: A listing of the path a species takes to reach 6174.

Properties of the Order: (1) The value of $a - d$,

(2) The value of $b - c$,

(3) The required number of iterations to reach 6174.

Figure 3 shows a graphical representation of these definitions.

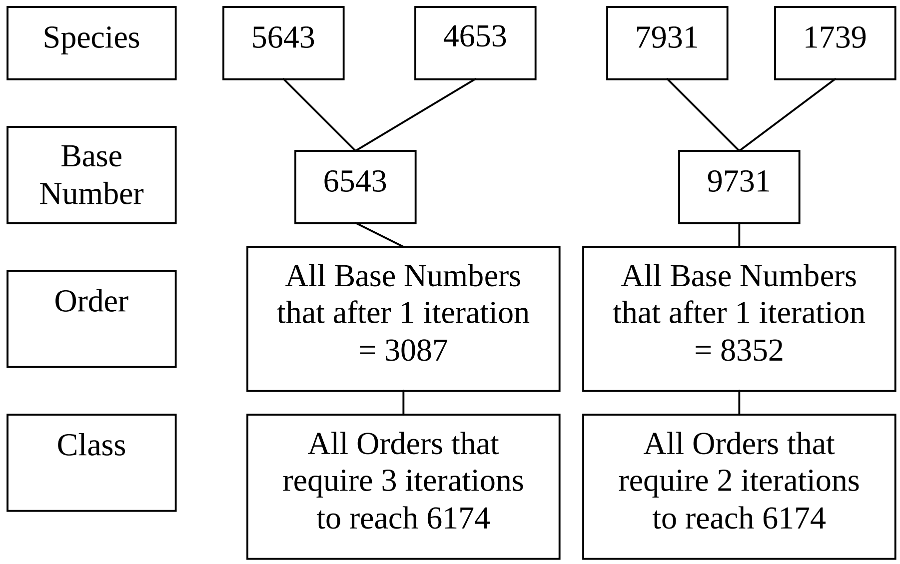


Figure 3

Now let us consider the family tree of the Kaprekar constant 6174 (See Appendix A for the Family Tree of all Four-Digit Numbers). Just as an individual's family tree may be used to trace that individual back to a particular ancestor, the family tree of a species may be used to trace it back to the common ancestor, 6174. We may also define the generation of a class to be the number of iterations separating it from this common ancestor, 6174.

In order to generate the family tree of this Kaprekar constant, we simply arranged all the species first by number of iterations, next by the difference *a-d*, and lastly by the difference *b-c*. In order to make this table more readable, we removed all the species, or the 0^{th} iteration, leaving us with all the unique paths of the 54 (or 55 including multiples of 1111) orders within the kingdom of four-digit numbers. Finally, in order to be able to better interpret our resulting graph, we color-coded it according to class.

3. Kaprekar Procedure with Five-Digit Numbers

Let us now look at what happens when you apply the Kaprekar Process to five-digit numbers. After testing all 100,000 five-digit numbers, we found that any five-digit number, such that all five digits are not the same, will enter one of three cycles. The first cycle, {53955, 59994}, will be entered within two iterations of the Kaprekar Process. The other two cycles, {71973, 83952, 74943, 62964}, {75933, 63954, 61974, 82962}, will be entered within six iterations of the Kaprekar Process. The five-digit numbers can enter one of these three cycles at any point in the cycle. We have yet to discover a way to predict what cycle a number will enter, or how many iterations are required for a number to enter one of these cycles. We have found that 3% of the five-digit numbers will enter the first cycle, 48% will enter the second cycle, and 49% will enter the last cycle. We have found that it is much more difficult to predict the behavior of the numbers that enter cycles, than it is to predict the behavior of the numbers that terminate.

4. Kaprekar Procedure with Six-Digit Numbers

When applied to six-digit numbers, the Kaprekar Process will do one of three things. It will either terminate with the number 549945 upon the first iteration, terminate with the number 631764 within the first four iterations, or enter a seven-element cycle within the first 13 iterations. Only 17 classes of six-digit numbers reach one of the above two terminal numbers. Of these 17 cases, all but one of them terminates with the number 631764. All the other classes enter the cycle {642654, 420876, 851742, 750843, 840852, 860832, 862632}.

The following charts (arranged by the value of *a-f*, where the six-digit number, with its digits in decreasing order, is represented by *abcdef*) indicate the behaviors of these various classes of six-digit numbers. The entries in the chart represent the number of iterations required to reach either the cycle or the terminating number. Asterisks are used to indicate classes of numbers terminating at 631764, and a circle is used to indicate the class of numbers terminating at 549945.

a-f= 0

b-e \ *c-d*	0
0	---

a-f=1

b-e \ *c-d*	0	1
1	6	5
0	11	---

a-f=2

b-e \ *c-d*	0	1	2
2	6	5	8
1	9	5	---
0	10	---	---

a-f= 3

b-e \ *c-d*	0	1	2	3
3	10	9	4	6
2	8	4	7	---
1	6	5	---	---
0	9	---	---	---

a-f= 4

b-e \ *c-d*	0	1	2	3	4
4	8	7	3	2	6
3	4	12	2*	3*	---
2	6	1	4	---	---
1	6	8	---	---	---
0	5	---	---	---	---

a-*f* = 5

b-*e* \ *c*-*d*	0	1	2	3	4	5
5	1°	10	7	7	10	6
4	9	10	7	7	10	---
3	6	9	5	5	---	---
2	3	4	9	---	---	---
1	6	9	---	---	---	---
0	10	---	---	---	---	---

a-*f* = 6

b-*e* \ *c*-*d*	0	1	2	3	4	5	6
6	8	12	2*	3*	10	9	9
5	8	7	3	2	6	10	---
4	8	7	3	1	6	---	---
3	4	12	1*	3*	---	---	---
2	5	2	4	---	---	---	---
1	6	8	---	---	---	---	---
0	10	---	---	---	---	---	---

a-*f* = 7

b-*e* \ *c*-*d*	0	1	2	3	4	5	6	7
7	8	4	7	8	8	10	8	7
6	3*	9	2	6	2*	5	2*	---
5	3	1	8	10	2	7	---	---
4	3*	2	8	10	2	---	---	---
3	10	9	2	6	---	---	---	---
2	8	4	7	---	---	---	---	---
1	6	5	---	---	---	---	---	---
0	5	---	---	---	---	---	---	---

a-f= 8

b-e \ *c-d*	0	1	2	3	4	5	6	7	8
8	7	5	11	3*	3	3*	3	3*	11
7	8	6	7	7	4	9	4	7	---
6	7	3	11	2	2*	5	2*	---	---
5	3	4	1	8	3	7	---	---	---
4	7	4	2	8	3	---	---	---	---
3	8	3	11	2	---	---	---	---	---
2	7	6	8	---	---	---	---	---	---
1	9	5	---	---	---	---	---	---	---
0	9	---	---	---	---	---	---	---	---

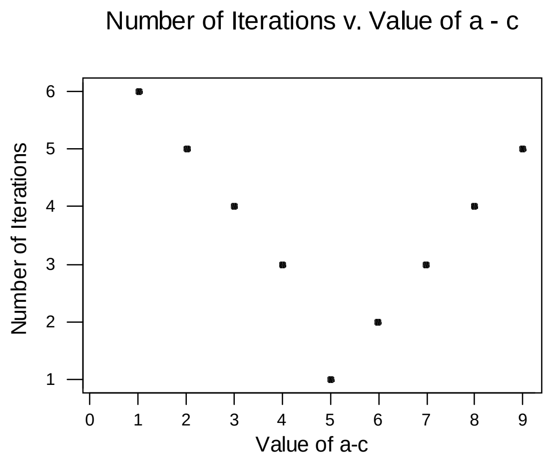
a-f= 9

b-e \ *c-d*	0	1	2	3	4	5	6	7	8	9
9	6	10	9	9	6	13	6	9	9	10
8	12	5	5	5	8	9	8	5	5	---
7	8	5	6	4	2	4	2	4	---	---
6	3*	5	3	9	12	9	12	---	---	---
5	8	4	5	5	7	10	---	---	---	---
4	3*	6	4	5	7	---	---	---	---	---
3	8	5	3	9	---	---	---	---	---	---
2	12	5	6	---	---	---	---	---	---	---
1	6	5	---	---	---	---	---	---	---	---
0	10	---	---	---	---	---	---	---	---	---

Notice that a great amount of symmetry exists in these charts. Many palindromes and mirror effects can be found throughout. The palindrome and mirror effects seem to be maximized at large values of *a-f*.

5. Kaprekar Process with Other Digit Lengths

It is interesting to note that for three-digit numbers, the Kaprekar Process will ultimately terminate with the number 495 after six or fewer iterations [3]. The number of iterations required to reach this number is determined directly by the difference *a-c*, a relationship portrayed by the following graph.



When we studied numbers of other digit lengths, we found that some numbers will terminate, while others will enter cycles. For some digit lengths (e.g. 3, 4), the numbers will only converge to a terminating number. Still for other digit lengths (e.g. 5, 7), the numbers will never terminate, but rather they will enter a cycle of some length. To determine the behavior of a number of a particular digit length, see Appendix B.

As one might imagine, it is rather difficult to determine which cycles a number will enter without experimenting with a number of elements in a given digit length, an operation that can be rather time-consuming, and, when done by hand, prone to error. On the other hand, it is quite easy to find terminating numbers by construction based upon already known terminating numbers of other digit lengths. For example, let us consider the number 6174, which is the exclusive terminal number for four-digit numbers. If we expand this number into six digits by inserting a ‘3’ between the ‘6’ and ‘1’ and a ‘6’ between the ‘7’ and ‘4,’ then we obtain the number 631764. This number may be recognized as one of the terminating numbers for six digits. If we repeat this process of insertion, we get the 8-digit number 63317664, which may be shown to be a terminal number as well. This holds indefinitely for all numbers with an even digit length with the exception of two. This may be shown as follows.

Arrange the number 633…331766…664 in ascending and descending order, and subtract the latter from the former.

$$\begin{array}{ccccccccccccccc}
 & & & & & & & 5 & 13 & 12 & 12 & & 12 & 12 & 11 \\
 & 7 & 6 & 6 & \dots & 6 & 6 & \cancel{6} & \cancel{4} & \cancel{3} & \cancel{3} & \dots & \cancel{3} & \cancel{3} & \cancel{1} \\
- & 1 & 3 & 3 & \dots & 3 & 3 & 4 & 6 & 6 & 6 & \dots & 6 & 6 & 7 \\
\hline
 & 6 & 3 & 3 & \dots & 3 & 3 & 1 & 7 & 6 & 6 & \dots & 6 & 6 & 4
\end{array}$$

Thus, we see that numbers with an even digit length (*n*) (with the exception of two) have at least one terminating number which may be found by inserting $\frac{n-4}{2}$ 6’s and $\frac{n-4}{2}$ 3’s into the “seed” number 6174.

By a similar process, we can find terminating numbers for numbers whose digit lengths are a multiple of three. This is done by expanding the three-digit terminating number 495. For six-digit numbers, for instance, when a-f = b-e = 5 and c-d = 0, the result of the Kaprekar Process is the terminating number 549945. Observe that this constant contains two 4’s, two 5’s, and two 9’s. Furthermore, for nine-digit numbers, the number 554999445 is a terminating number and contains three 4’s, three 5’s, and three 9’s. Again, we can show this to be true as follows.

					8	14	14		14	14	13	13		13	14
	9	9	…	9	9	5	5	…	5	5	4	4	…	4	4
-	4	4	…	4	4	5	5	…	5	5	9	9	…	9	9
	5	5	…	5	4	9	9	…	9	9	4	4	…	4	5

Thus, a terminating number can be found for any number containing $3k$ digits, $k \in \mathbb{Z}$.

A similar method of proof can be used for numbers whose digit lengths are multiples of eight and ten. All of these may be found on the summary statistics chart in Appendix B.

6. Conclusion

We have explored what happens when you apply the Kaprekar Process to three-, four-, five-, and six-digit numbers, and then generalized the results for other digit lengths. Our research has shown that four-digit numbers will terminate within seven iterations of the Kaprekar Process with the number 6174 [1]. Five-digit numbers will enter one of three cycles within six iterations of the Kaprekar Process. Within thirteen iterations, six-digit numbers will either terminate or will enter a cycle, and three-digit numbers will terminate with 495 [3]. We were able to predict how many iterations it will take any four-digit number to reach 6174, and how many iterations it would take any six-digit number to reach one of its terminal numbers. We have answered many questions, but we have discovered many new questions through our research of the Kaprekar Process. For example, why, on the four-digit number's iteration table, is the diagonal 7-1-6-2-7 not a palindrome? With six-digit numbers, why is there just one number that terminates at 549945? These and other questions need to be the subject of further research. Some of our results duplicate previously discovered facts, which we realized through a MathSciNet search, but much of our work is new. Some of these previously discovered facts are that a four-digit number will terminate within seven iterations with the number 6174 [1, 2], and a three-digit number will terminate with the number 495 [3].

We would like to thank Dr. Anant Godbole, who introduced us to this topic and spent many hours assisting us with our research over the course of three semesters. In addition, we would like to thank the referee for making so many helpful suggestions.

REFERENCES

1. D. R. Kaprekar, *An Interesting Property of the Number 6174*, Scripta Math., 21(1955) 304.
2. ----------, *The New Constant 6174*, Devlali Camp, Devlali, India, 42 pages, paper bound.
3. Klaus E. Eldridge and Seok Sagong, *The Determination of Kaprekar Convergence and Loop Convergence of All Three-Digit Numbers*, American Mathematical Monthly, 95(1988), 105-112.

Appendix A

Number of Iteration	Value of a - d	Value of b - c	Iteration 1	Iteration 2	Iteration 3	Iteration 4	Iteration 5	Iteration 6	Iteration 7
0	0	0							
1	6	2	6174						
2	4	2	4176	6174					
2	8	4	8352	6174					
2	8	6	8532	6174					
3	2	1	2088	8532	6174				
3	3	1	3087	8352	6174				
3	4	3	4266	4176	6174				
3	6	3	6264	4176	6174				
3	7	1	7083	8352	6174				
3	7	4	7353	4176	6174				
3	7	6	7533	4176	6174				
3	8	1	8082	8532	6174				
3	9	2	9171	8532	6174				
3	9	3	9261	8352	6174				
3	9	7	9621	8352	6174				
3	9	8	9711	8532	6174				
4	1	1	1089	9621	8352	6174			
4	2	0	1998	8082	8532	6174			
4	4	0	3996	6264	4176	6174			
4	4	4	4356	3087	8352	6174			
4	6	4	6354	3087	8352	6174			

4	6	6	6534	3087	8352	6174		
4	7	0	6993	6264	4176	6174		
4	9	0	8991	8082	8532	6174		
4	9	1	9081	9621	8352	6174		
4	9	9	9801	9621	8352	6174		
5	1	0	0999	8991	8082	8532	6174	
5	3	2	3177	6354	3087	8352	6174	
5	5	3	5265	3996	6264	4176	6174	
5	5	4	5355	1998	8082	8532	6174	
5	5	5	5445	1089	9621	8352	6174	
5	6	5	6444	1998	8082	8532	6174	
5	7	2	7173	6354	3087	8352	6174	
5	7	5	7443	3996	6264	4176	6174	
5	8	3	8262	6354	3087	8352	6174	
5	8	7	8622	6354	3087	8352	6174	
6	2	2	2178	7443	3996	6264	4176	6174
6	3	0	2997	7173	6354	3087	8352	6174
6	3	3	3267	5265	3996	6264	4176	6174
6	5	0	4995	5355	1998	8082	8532	6174
6	6	0	5994	5355	1998	8082	8532	6174
6	7	3	7263	5265	3996	6264	4176	6174
6	7	7	7623	5265	3996	6264	4176	6174
6	8	0	7992	7173	6354	3087	8352	6174
6	8	2	8172	7443	3996	6264	4176	6174
6	8	8	8712	7443	3996	6264	4176	6174

6174	6174	6174	6174	6174	6174	6174	6174
4176	8352	8532	4176	8532	4176	8352	4176
6264	3087	8082	6264	8082	6264	3087	6264
3996	6354	1998	3996	1998	3996	6354	3996
7443	7173	5355	7443	5355	7443	7173	7443
8172	7992	5994	8172	5994	8172	7992	8172
4086	5085	5175	6084	8442	9351	9441	9531
1	1	2	1	5	4	5	6
4	5	5	6	8	9	9	9
7	7	7	7	7	7	7	7

Appendix B

Number of Digits	Terminations		Unique Loops	
	Terminating numbers	Number of Iterations to Terminate	Size of Unique Loops	Number of Iterations to Enter Loops
2	—	—	5	0-2
3	495	0-6	—	—
4	6174	0-7	—	—
5	—	—	2 4 4	0-2 0-6 0-6
6	549945 631764	0-1 0-4	7	0-13
7	—	—	8	0-13
8	63317664 97508421	0-4 0-3	7 3	0-15 0-19
9	554999445 864197532			
10	6333176664 9753086421			
11	86431976532			
12	633331766664			

	555499994445 975330866421			
13	8643319766532			
14	63333317666664 97533308666421			
15	555549999944445 864333197666532			
16	6333333176666664 9977551088442201			
17	86433331976666532			
18	633333331766666664 886644219977553312 555554999999444445			
19	8643333319766666532			
20	63333333317666666664 99775533108866442201			
21	555555499999994444445			
22	6333333333176666666664			
23	86433333331976666666532			
24	555555549999999944444445 633333333333176666666664 999777555110888444222001			
25	8643333333331976666666532			

26	6333333333331766666666666664			
27	555555554999999999444444445 888666444221999777555333112			